

# Multimode circuit QED with hybrid metamaterial transmission lines

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Quantum transmission lines are a central to superconducting and hybrid quantum computing[1]. Parallel to these developments are those of left-handed meta-materials. They have a wide variety of applications in photonics from the microwave to the visible range such as invisibility cloaks and perfect flat lenses [2, 3]. For classical guided microwaves, left-handed transmission lines have been proposed [4] and studied [5] on the macroscopic scale. We combine these ideas in presenting a left-handed/right-handed hybrid transmission line for applications in quantum optics on a chip. The resulting system allows circuit QED to reach a new regime: multi-mode ultra-strong coupling. Out of the many potential applications of this novel device, we discuss two; the preparation of multipartite entangled states and its use as a quantum simulator for the spin-boson model[6] where a quantum phase transition is reached up to finite size-effects.

## I. INTRODUCTION

Quantum optics addresses the interaction of quanta of matter — atoms — with quanta of electromagnetic fields — photons. This is beautifully realized in cavity quantum electrodynamics (QED), where the interaction between those units is made strong by confining the field into a small mode volume [7]. Circuit QED takes this further by confining microwave photons in a quasi 1D strip-line cavity and using superconducting qubits as *artificial* atoms with a large dipole moment [1, 8]. Next to being a promising architecture for quantum computing, a multitude of basic quantum optical effects has been demonstrated [9]. Going beyond what can be reached in atomic systems, an ultrastrong coupling regime — where the coupling strength becomes comparable to the atomic energy scales — has been proposed [10] and achieved [11, 12]. Furthermore in the circuit QED approach, elements are entirely human-made and can hence be flexibly engineered. This can lead to coupling to multiple modes [13–17] either in the same or distinct cavities. There is a wealth of proposals exploiting these features to create complex photonic states [18–20] involving a large number of cavities.

## II. THE SYSTEM

In one-dimension, left-handedness is defined as the wave vector  $\mathbf{k}_l$  and the Poynting vector having opposite orientation; the phase and group-velocity are opposite. The triplet of electric field, magnetic field, and wave vector,  $(\mathbf{E}, \mathbf{H}, \mathbf{k}_l)$ , is left-handed yielding a falling dispersion relation  $\partial\omega(k)/\partial k < 0$ . This can be achieved[4] by a discrete array of series capacitors and parallel inductors to ground, see Fig. 1. A low loss left-handed transmission line (LHTL) can be realized with superconductors [5, 21]. This is the dual (inductors and capacitors inter-

changed) of the usual [22] discrete representation of the right-handed transmission line (RHTL). In practice, the LHTL remains a metamaterial composed of discrete elements, whereas the RHTL is a metal strip represented as the continuum limit of a ladder network [22]. Unusual physics arises when right- and left-handed media are interfaced [2]. We realise this with a coupled transmission line (CTL) shown in Fig. 1, a discrete LHTL coupled to a RHTL to be taken into the continuum limit. This model is studied using Kirchhoff laws with a plane wave ansatz, details are in the supplementary material. Individually, each component has the dispersion relation

$$\begin{aligned}\omega_r(k_r) &= \frac{2}{\sqrt{C_r L_r}} \sin\left(\frac{k_r \Delta x}{2}\right) \xrightarrow{\text{continuum}} \frac{k_r}{\sqrt{c_r l_r}} \\ \omega_l(k_l) &= \frac{1}{2\sqrt{C_l L_l} \sin\left(\frac{k_l \Delta x}{2}\right)}\end{aligned}\quad (1)$$

where  $\omega_{r/l}$  are the angular frequencies of the RHTL/LHTL respectively.  $C_{l/r}$  and  $L_{l/r}$  are capacitances and inductances as defined in Fig. 1 and  $c_r$  and  $l_r$  are the capacitance and inductance per unit length in the RHTL.  $\Delta x$  is the size of a unit cell. A key unusual feature of the LHTL, as compared to a regular RHTL, is the divergence of the density of modes (DoM) at a low-frequency bound  $\omega_{\text{IR}} = 1/2\sqrt{C_l L_l}$ , seen in Fig. 2(a), implying the existence of a quasi-continuous band even in a cavity. In the LHTL, low frequencies correspond to short wavelengths due to the falling dispersion relation  $\omega(k)$ . Thus, by only a small change in frequency, a new orthogonal mode can be found that is different by one node in the left-handed component. As the wavelength approaches the lattice constant, the dispersion relation in equation (1) becomes flat due to Bragg reflection[23] — the DoM develops a van-Hove-type singularity setting the aforementioned divergence at  $\omega_{\text{IR}}$ . Due to the hybrid nature of this new CTL, the closely spaced frequencies at this lower band-edge have nearly-identical spatial structures in the RHTL. The fast oscillation in the LHTL

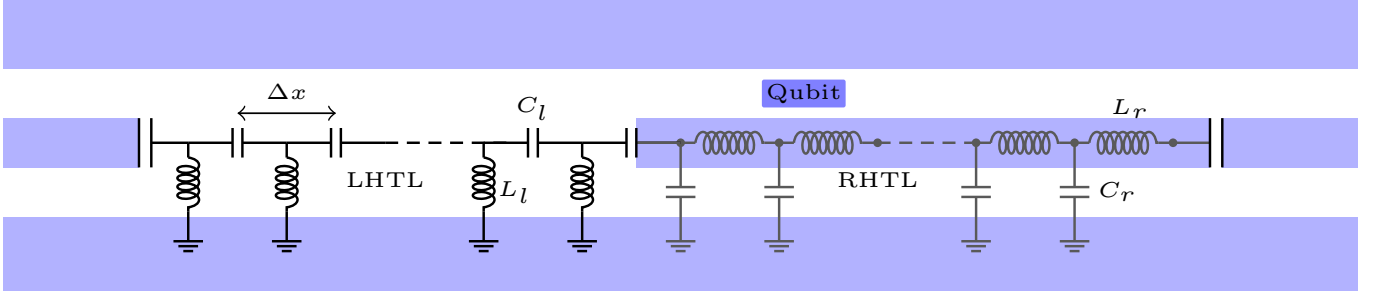


FIG. 1. Discrete LHTL coupled to a continuous RHTL. The regular right-handed part of the transmission line is on the right, connected to a left-handed line shown on the left. The terminating capacitors allow to externally access the modes. The qubit couples to the right-handed component. In the supplementary material it is shown that stray capacitance to ground in the left handed line does not change the physics. Additional stray inductances and capacitances can be taken into account as shown by Eleftheriades, *et al.* [4]. The light blue areas indicate strip-lines and ground planes.

ensure orthogonality between modes. Figure 2(b) shows three consecutive low frequency modes obtained from the full solution. The DoM can be approximated by the sum of the densities in the uncoupled lines. Thus the DoM of the CTL is approximately

$$\mathcal{D}(\omega) = \frac{4N_l\sqrt{C_l L_l}}{\pi} \tan \phi_L \sin \phi_L, \quad \phi_L = \frac{k_l(\omega)\Delta x}{2}$$

$N_l$  is the number of cells in the LHTL. The agreement between this prediction and the numerically obtained modes of the full model is excellent up to small oscillations, see Figure 2(a).

The quantum behaviour of the CTL is obtained through canonical quantization of the circuit in Fig. 1. This leads to a system of uncoupled quantum harmonic oscillators, each described by operators  $\hat{a}_n^\dagger, \hat{a}_n$  acting on modes with frequencies  $\omega_n$ . A qubit described in its energy eigenbasis by Pauli matrices  $\hat{\sigma}_{x/z}$  placed close to the CTL will couple to mode  $n$  with strength  $g_n$

$$\hat{H}/\hbar = \frac{\Delta_0}{2} \hat{\sigma}_z + \sum_n g_n \hat{\sigma}_x (\hat{a}_n + \hat{a}_n^\dagger) + \sum_n \omega_n \hat{a}_n^\dagger \hat{a}_n. \quad (2)$$

If the qubit is coupled to the RHTL,  $g_n \simeq g_{n+1}$  for low frequency modes since they have similar spatial profiles in the RHTL. For a flux qubit [24, 25], the mode dependent part of the coupling strength is given by  $\mathcal{D}(\omega_n) \langle I_n(x) \rangle / \max_n \{ \langle I_n(x) \rangle \}$ . The current  $I_n$  is averaged over the spatial extend of the qubit. Figure 2(a) shows that the qubit can be coupled to a wide range of modes. This mode structure allows the qubit to simultaneously couple to multiple modes when  $N > 1$  modes fall within a frequency interval of  $2g_n$ . We refer to this regime as *multi-mode strong-coupling*. It can be reached with other qubits, notably transmons [26]. However, flux qubits allow us to reach *multi-mode ultrastrong coupling* [10, 11] —  $g_n/\omega_n > 0.1$ . This regime offers many new possibilities for circuit QED.

### III. APPLICATIONS

The multimode Rabi Hamiltonian, equation (2), allows us to prepare multimode entangled states. Within the rotating wave approximation it conserves the number of excitations. Exciting the qubit and placing its  $0 \leftrightarrow 1$  resonance frequency slightly above  $\omega_{\text{IR}}$  allows the qubit excitation to distribute itself over many modes, i.e., produce arbitrary superpositions of the form  $c_0|1; 0\rangle + \sum_n c_n|0; n\rangle$ .  $|0; n\rangle$  indicates the qubit in the ground state, a single photon in mode  $n$  and none in the other modes. For  $|1; 0\rangle$  only the qubit is excited. These states are in general entangled as seen from their Von Neumann entropy [27]. Figure 3 shows the entropy, indicating multimode entanglement, of a single excitation, starting in  $|1; 0\rangle$ , that spread out over all modes for a time  $t$ .

The Spin-Boson model [6] is a fundamental model of quantum dissipation which allows to understand the transition between coherent and incoherent behaviour as well as a quantum phase transition suppressing quantum tunnelling. It is described by the Hamiltonian in equation (2) in the limit where the modes form a continuum. The dense modes at the low-frequency end provide a generic and realizable quantum simulator for this model. Our unusual density of modes provides a novel regime of sub-subhomic models with a low-frequency cutoff, i.e., a spectral density of the form

$$J(\omega) = \sum_n g_n^2 \delta(\omega - \omega_n) \simeq \frac{N_l}{\pi \sqrt{2\omega_{\text{IR}}}} \frac{\Theta(\omega - \omega_{\text{IR}})}{\sqrt{\omega - \omega_{\text{IR}}}}.$$

The ground and lowest excited states are well approximated by a multimode Schrödinger cat state of the qubit dressed by coherent photonic states

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left( |L\rangle \bigotimes_n |\lambda_n\rangle \pm |R\rangle \bigotimes_n |-\lambda_n\rangle \right). \quad (3)$$

$|L, R\rangle$  are the eigenstates of  $\hat{\sigma}_x$ . The renormalized energy

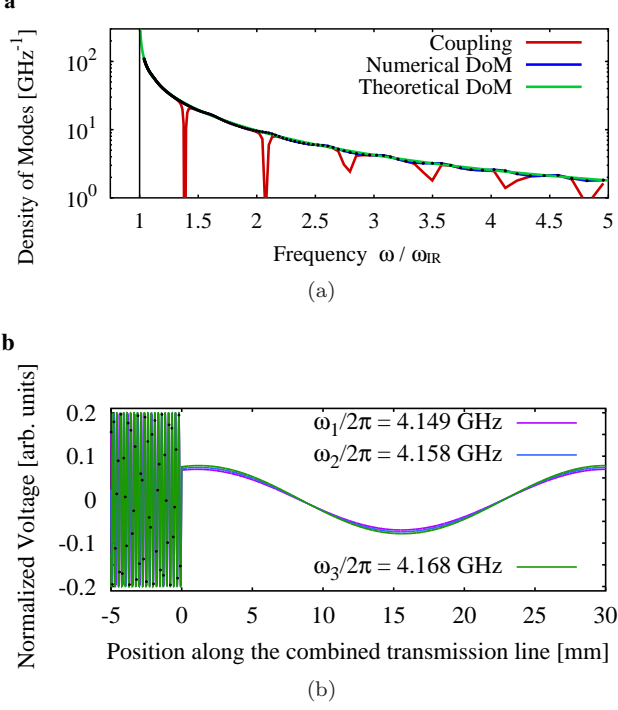


FIG. 2. (a) Example of density of modes, showing a lower band-edge at  $\omega_{\text{IR}}$ . Dots indicate actual modes, the blue curve the approximate formula which is in excellent agreement. The red curve shows the coupling strength between a 0.5 mm long flux qubit placed at a current anti-node of the transmission line, as in Fig. 1. Designing the qubit to couple to the 4.751 GHz, mode with strength 200 MHz, results in ultrastrong-coupling to 36 modes within a  $\pm 200$  MHz range. (b) Example of the first three consecutive normal modes, the voltage profiles in the RHTL are almost identical. The LHTL has 200 unit cells measuring  $100 \mu\text{m}$  each. In the LHTL the voltages at the discrete unit cells, for the first mode only, are shown by the black dots; the continuous lines serve as a guide to the eye to see the mode structure. Requiring a  $50 \Omega$  impedance and an IR cutoff at  $\omega_{\text{IR}}/2\pi = 4$  GHz sets  $C_l = 398$  fF and  $L_l = 995$  pH. The parameters for the RHTL were chosen so that it supports a full wavelength at  $\omega_{\text{IR}}$ . This sets the values for its total inductance and capacitance. Therefore a 3 cm long RHTL requires a capacitance and inductance per unit length of  $c_r = 1667$  fF/ $\mu\text{m}$  and  $l_r = 4167$  pH/ $\mu\text{m}$ .

splitting is

$$\Delta_{\text{eff}} = \langle + | \hat{H} | - \rangle = \Delta_0 \exp \left( -2 \sum_n \lambda_n^2 \right). \quad (4)$$

In order for the multimode cat state in equation (3) to be a good approximation to the true ground state we apply adiabatic renormalization [6]. This suggests that fast modes, those with  $\omega_n > \Delta_{\text{eff}}$ , adiabatically follow the qubit. Slow modes remain unaffected. Thus  $\lambda_n = \frac{g_n^2}{\omega_n^2} \Theta(\omega_n - \Delta_{\text{eff}})$  which leads to a self-consistency relation for  $\Delta_{\text{eff}}$ . The ratio  $\Delta_{\text{eff}}/\Delta_0$  measures the accumulated phase space distance of the dressing clouds, i.e.,

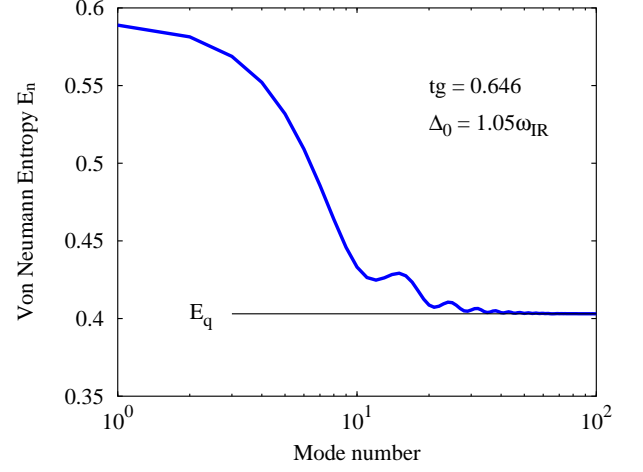


FIG. 3. Von Neumann entropy as function of the traced out mode  $n$ . Here the qubit is put in the bath at a frequency  $\omega_q > \omega_{\text{IR}}$  and the system left to evolve for a dimensionless time  $tg$ . The Von Neumann entropy is computed for the system after having traced out the qubit and mode  $n$ . We then vary  $n$  to see how much the entropy increases from  $E_q$ , the entropy when only the qubit is traced out.  $E_q > 0$  indicates that there is at least bi-partite entanglement. When additionally tracing out mode  $n$  the entropy  $E_n$  increases above  $E_q$  and this for all modes. The latter indicates complex multi-partite entanglement.

the total cat size [7] by taking the logarithm of equation (4). Thus, the low-energy states of the system are strongly renormalized as are the effective energy scales.

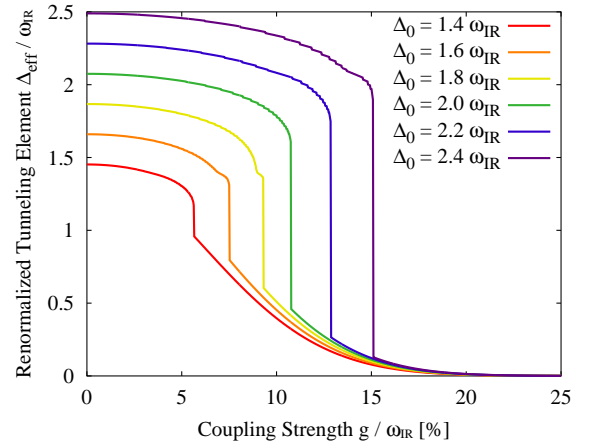


FIG. 4. Plot of the renormalized tunnelling as function of the global coupling strength  $g$ . The individual mode coupling  $g_n$  is the product of  $g$  and the mode-dependent spatial contribution bounded by unity. This allows simultaneous tuning of the coupling to all modes. The discontinuous drop in  $\Delta_{\text{eff}}$  distinguishes the regions of weakly and strongly renormalized energy splitting.

A true dissipative quantum phase transition [6, 28] has  $\Delta_{\text{eff}} = 0$  in the localised phase. This limit would be

reached if the modes were infinitely close (hence arbitrarily close to  $\omega_{\text{IR}}$ ) as would result from an infinitely long LHTL or if  $\omega_{\text{IR}} \rightarrow 0$  as in the case of infinitely dense LHTL unit cells. Note that in the usual sub-Ohmic spin-boson model, the latter is assumed. We thus conclude that our system approaches a quantum phase transition in the infinite sample limit.

To corroborate the finite size-behaviour, we have studied the ground and first excited state of the qubit-CTL model using its actual modes in the adiabatic renormalization approach. We identify multiple regimes: for weak coupling or large  $\Delta_0/\omega_{\text{IR}}$ , there is only weak dressing manifest by a small shift of  $\Delta_0$ . At stronger coupling, we observe the quasi-localized phase, with  $\Delta_{\text{eff}} \ll \Delta_0$ . Remarkably, even at finite length, the two regimes are separated by a discontinuous transition as indicated by Figure 4. Figure 5 shows the corresponding finite-size phase diagram highlighting the need for ultrastrong coupling. We see that by tuning the bare qubit frequency slightly above the cutoff,  $\Delta_0 > \omega_{\text{IR}}$  [29] we can tune the system through the phase transition *in situ*.

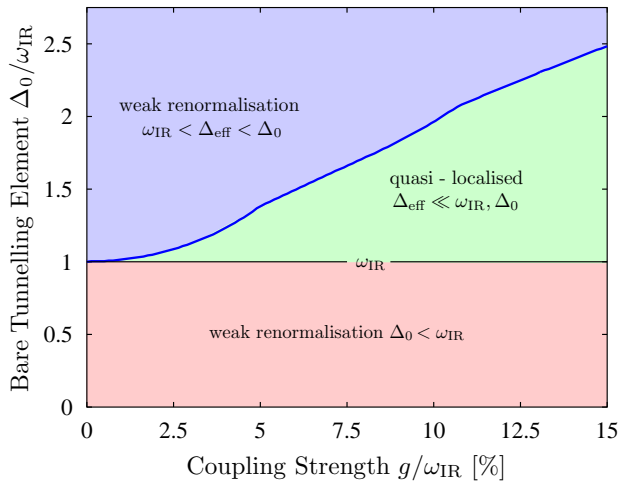


FIG. 5. Finite-size phase diagram of the qubit-CTL system. The density of states for the case at hand is the same as in Fig. 2(a).

On the level of partition functions, this model is equivalent to a one-dimensional Ising chain [28, 30]. This is discussed in more detail in the supplementary material. In the present case, this would be an Ising chain with an interaction that decays as  $\propto |i - j|^{-1/2}$ , where  $i$  and  $j$  are site indices, up to a range  $r \propto \omega_{\text{IR}}^{-1}$ , after which it decays exponentially. Thus, when cooled from high-temperatures the system is well described by mean-field theory, which predicts a ferromagnetic phase transition, until the correlation length reaches  $r$ . At that point, the system follows short-range physics and remains paramagnetic between magnetized blocks of size  $r$  — in analogy to the tunnel coupling in the spin-boson model falling deeply, but not to zero.

## IV. CONCLUSION

We have proposed an engineered hybrid transmission line that allows to reach a new multimode strong coupling regime of circuit QED by combining a regular line with a metamaterial. This will open the way for novel applications in microwave photonics and strongly correlated photon states, out of which we have outlined the generation of multimode entanglement, multimode Schrödinger cat states and quantum phase transitions.

## V. ACKNOWLEDGEMENTS

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### Appendix A: Electric Properties of Right and Left-Handed Transmission lines

The following section presents the electric properties of the composite transmission line. The derivations follow typical microwave engineering as outlined in Chapter 2 of Ref. 22. We show how the system behaves at the boundary, i.e. under what conditions energy flow across the boundary is optimized. Examining the relation between voltage and current in each medium gives the impedance of the lines; the latter are then used to express the reflection and transmission coefficients at the boundary. They are found by imposing continuity of voltage and current. Then the voltage profile along the CTL is discussed. The section finishes in showing that there is a dense mode spectrum at a low-frequency bound.

#### 1. Equations of Motion

The right and left-handed parts of the transmission line are respectively shown in the right and left side of Figure 1 of the main text. In terms of magnetic flux  $\Phi$  their uncoupled Lagrangians [1, 31] are

$$\mathcal{L}_r = \sum_{n < 0} \left[ \frac{1}{2} C_r \dot{\Phi}_n^2 - \frac{1}{2L_r} (\Phi_n - \Phi_{n-1})^2 \right]$$

$$\mathcal{L}_l = \sum_{n > 0} \left[ \frac{1}{2} C_l (\dot{\Phi}_n - \dot{\Phi}_{n+1})^2 - \frac{1}{2L_l} \Phi_n^2 \right]$$

$C_r$ ,  $L_r$  are the capacitance and inductance of the discrete model of the RHTL which will then be taken to the continuum limit.  $C_l$  and  $L_l$  are the capacitance and inductance of a unit cell of the LHTL. Through the Euler-Lagrange equation, these Lagrangians produce the fol-

lowing equations of motion

$$\frac{1}{C_r L_r} (\Phi_{n-1} - 2\Phi_n + \Phi_{n+1}) = \ddot{\Phi}_n \quad (\text{A1})$$

$$C_l L_l (\ddot{\Phi}_{n-1} - 2\ddot{\Phi}_n + \ddot{\Phi}_{n+1}) = \Phi_n \quad (\text{A2})$$

The latter are none other than Kirchoff's current law for each cell of the transmission line. This is seen by noting that, for the case of the RHTL, the voltage on capacitor  $n$  is  $V_n = -\dot{\Phi}_n$  and the current flowing through this capacitor is  $I_{C_n} = C_r \dot{V}_n$ . The flux difference between two nodes relates to current through  $\Phi_n - \Phi_{n-1} = L_r I_{n-1}$ . Thus equation (A1) is none other than  $I_n - I_{n-1} = I_{C_n}$ , see also figs. 6(a),6(b). Similar reasoning applies to the LHTL.

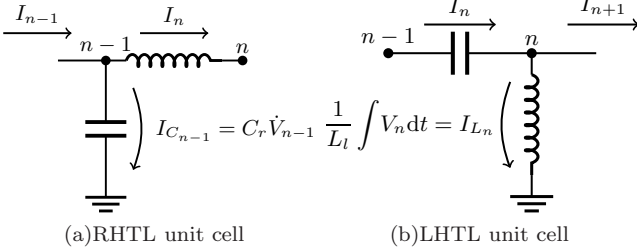


FIG. 6. Unit cells of both transmission line with Kirchoff's current law.

## 2. Dispersion Relation and Wavelength

The dispersion relations in the discrete RHTL and LHTL are found from the equations of motion A1 by assuming a plane wave ansatz  $\Phi_n(t) = \exp\{-i(kn\Delta x - \omega t)\}$  and using the relation  $2 - e^{-ik\Delta x} - e^{ik\Delta x} = 4 \sin^2(k\Delta x/2)$ .

$$\omega_r(k_r) = \frac{2}{\sqrt{C_r L_r}} \sin\left(\frac{k_r \Delta x}{2}\right) \quad (\text{A3})$$

$$\omega_l(k_l) = \frac{1}{2\sqrt{C_l L_l}} \frac{1}{\sin\left(\frac{k_l \Delta x}{2}\right)} \quad (\text{A4})$$

$\Delta x$  is the length of a unit cell.  $\omega_{r/l}$  is the angular frequency of a wave with wavevector  $k_{r/l}$  in the RHTL/LHTL. In the right-handed medium, the usual continuous dispersion relations  $vk = \omega$  can be recovered by taking the limit  $\Delta x \rightarrow 0$ . With the inductance and capacitance per unit-length  $l_r = L_r/\Delta x$  and  $c_r = C_r/\Delta x$  respectively, the square of equation (A3) becomes

$$\omega^2 l_r c_r (\Delta x)^2 = 2 - 2 \cos(k\Delta x) \approx (k\Delta x)^2$$

Cancelling  $(\Delta x)^2$  on both sides and identifying  $v = 1/\sqrt{l_r c_r}$  as the phase velocity of the waves gives the usual continuous dispersion relation for the RHTL.

## 3. Brillouin Zone of the Transmission Lines

Given that the transmission line is made of discrete elements of length  $\Delta x$  the minimum distinguishable wavelength is  $\lambda_{\min} = 2\Delta x$ . In the case where both lines are made of discrete elements there is both an IR and UV cutoff. In the case where the RHTL is continuous, the UV cutoff is set by the superconducting gap of the material, this shall be ignored given that it happens at frequency scales that are not relevant when coupling qubits to transmission lines. Measuring distance in units of the unit cells, i.e.  $\Delta x = 1$ , gives the maximum allowed wave number as

$$k_{\max} = \frac{2\pi}{\lambda_{\min}} = \pi$$

In the left-handed medium an upper bound on the maximum wave number corresponds to a lower bound on the frequency since  $k_l \sim 1/\omega_l$ . In the right-handed medium  $k_{\max}$  gives an upper bound on the frequency

$$\omega_{\text{IR}} = \frac{1}{2\sqrt{C_l L_l} \sin\left(\frac{k_{\max}}{2}\right)} = \frac{1}{2\sqrt{C_l L_l}} \quad (\text{A5})$$

$$\omega_{\text{UV}} = \frac{2}{\sqrt{C_r L_r}} \sin\left(\frac{k_{\max}}{2}\right) = \frac{2}{\sqrt{C_r L_r}} \quad (\text{A6})$$

Therefore due to the discreteness of the composite TL, the allowed modes fall in the interval  $[\omega_{\text{IR}}, \omega_{\text{UV}}]$  — the Brillouin zone of the discrete CTL. In what follows, it shall be assumed that the RHTL is continuous and has no upper frequency bound other than the superconducting gap.

## 4. Current, Voltage and Impedance

In the following section the relation between current and voltage is examined in order to find the impedance. Making use of Kirchoff's laws and considering the RHTL's unit cell shown in Fig. 6(a) gives

$$I_{n-1} - I_n = I_{C_{n-1}} = C_r \dot{V}_{n-1} = -i\omega C_r V_{n-1} \quad (\text{A7})$$

In the LH unit cell, shown in Fig. 6(b), current and voltage are related through

$$I_n - I_{n+1} = I_{L_n} = \frac{1}{L_l} \int V_n dt = \frac{i}{\omega L_l} V_n \quad (\text{A8})$$

### a. Plane Wave Ansatz

The voltage along the CTL can be decomposed into two parts, each originating in one side of the CTL and propagating towards the boundary. The waves propagating along the CTL must have the same energy therefore we set  $\omega_l = \omega_r = \omega$ .  $V^+$  originates in the RHTL whilst



$V^-$  comes from the LHTL. Explicitly, these two dimensionless voltages with unity as maximum amplitude are

$$V_n^+(t) = e^{-i\omega t} \begin{cases} e^{-ik_r n \Delta x} + \Gamma^+ e^{ik_r n \Delta x} & n \geq 0 \\ T^+ e^{ik_l n \Delta x} & n \leq 0 \end{cases} \quad (\text{A9})$$

$$V_n^-(t) = e^{-i\omega t} \begin{cases} T^- e^{ik_r n \Delta x} & n \geq 0 \\ e^{-ik_l n \Delta x} + \Gamma^- e^{ik_l n \Delta x} & n \leq 0 \end{cases} \quad (\text{A10})$$

When the waves encounter the interface at  $n = 0$  they are partially transmitted and reflected;  $\Gamma^\pm$  and  $T^\pm$  are the reflection and transmission coefficients for  $V^\pm$ ; they will be determined in section A 5. It is important to note that in the left-handed line the group velocity is negative: the sign of the wave vector is opposite to the Poynting vector. The voltages given by (A9) and (A10) are associated with currents that take the form

$$I_n^+(t) = e^{-i\omega t} \begin{cases} I_0^+ e^{-ik_r n \Delta x} + I_\Gamma^+ \Gamma^+ e^{ik_r n \Delta x} & n \leq 0 \\ I_T^+ T^+ e^{ik_l n \Delta x} & n \geq 0 \end{cases} \quad (\text{A11})$$

$$I_n^-(t) = e^{-i\omega t} \begin{cases} I_T^- T^- e^{ik_r n \Delta x} & n \leq 0 \\ I_0^- e^{-ik_l n \Delta x} + I_\Gamma^- \Gamma^- e^{ik_l n \Delta x} & n \geq 0 \end{cases} \quad (\text{A12})$$

The six coefficients  $I_0^\pm$ ,  $I_T^\pm$  and  $I_\Gamma^\pm$  are introduced to relate voltage to current. They are found by making use of Kirchoff's laws (A7) and (A8). For instance as an explicit example  $I_T^+$  is found by considering the current difference between cell  $n$  and  $n + 1$

$$\begin{aligned} I_n^+ - I_{n+1}^+ &= I_T^+ T^+ (1 - e^{-ik_l \Delta x}) e^{-i(k_l n \Delta x + \omega t)} \\ &= \frac{i}{\omega L_l} T^+ e^{-i(k_l n \Delta x + \omega t)} \end{aligned} \quad (\text{A13})$$

Where the first equality in (A13) follows from the plan wave ansatz and the second one is a result of Kirchoff's law. The six coefficients are then

$$I_0^+ = \frac{i\omega C_r}{e^{ik_r \Delta x} - 1} \quad (\text{A14})$$

$$I_0^- = \frac{i}{\omega L_l (1 - e^{ik_l \Delta x})} \quad (\text{A15})$$

$$I_T^- = I_\Gamma^+ = \frac{i\omega C_r}{e^{-ik_r \Delta x} - 1} \quad (\text{A16})$$

$$I_\Gamma^- = I_T^+ = \frac{i}{\omega L_l (1 - e^{-ik_l \Delta x})} \quad (\text{A17})$$

Relations (A14) and (A16) were found with Kirchoff's current law for the RHTL and (A15) and (A17) with the LHTL current law, i.e. equations (A7) and (A8) respectively.

## b. Transmission Line Impedance

Following [22] the impedances of the RHTL and LHTL,  $Z_r$  and  $Z_l$  respectively, are

$$\begin{aligned} Z_r &= -\frac{V_n^-}{I_n^-} = -\frac{e^{-ik_r \Delta x} - 1}{i\omega C_r} \xrightarrow{\Delta x \rightarrow 0} \sqrt{\frac{L_r}{C_r}} \\ Z_l &= \frac{V_n^+}{I_n^+} = -i\omega L_l (1 - e^{-ik_l \Delta x}) \\ &= \sqrt{\frac{L_l}{C_l} \left(1 - \frac{\omega_{\text{IR}}^2}{\omega^2}\right)} - \frac{i}{2\omega C_l} \end{aligned} \quad (\text{A18})$$

When taking the limit, the dispersion relation was used and the usual form for the impedance of a continuous strip line is recovered. Here the definition of  $\omega_{\text{IR}}$ , i.e. equation (A5) was used. This shows that the quantity under the square root is always positive. Furthermore the impedances enjoy the property that  $|Z_r| = \sqrt{L_r/C_r}$  and  $|Z_l| = \sqrt{L_l/C_l}$ . The terms featuring a frequency dependency are the result of the discreteness of the model. This can further be confirmed by taking the long wavelength limit; when  $\lambda \gg \Delta x$  it is expected that the discrete transmission lines behave like continuous ones. For the LHTL the long wavelengths are achieved by the limit  $\omega \rightarrow \infty$  for which  $Z_l \rightarrow \sqrt{L_l/C_l}$ . Making use of the impedance, the currents in relations (A11) and (A12) can be more conveniently written as

$$I_n^+(t) = e^{-i\omega t} \begin{cases} \frac{1}{Z_r^*} e^{-ik_r n \Delta x} - \frac{\Gamma^+}{Z_r} e^{ik_r n \Delta x} & n \leq 0 \\ \frac{T^+}{Z_l} e^{ik_l n \Delta x} & n \geq 0 \end{cases} \quad (\text{A19})$$

$$I_n^-(t) = e^{-i\omega t} \begin{cases} -\frac{T^-}{Z_r} e^{ik_r n \Delta x} & n \leq 0 \\ -\frac{1}{Z_l^*} e^{-ik_l n \Delta x} + \frac{\Gamma^-}{Z_l} e^{ik_l n \Delta x} & n \geq 0 \end{cases} \quad (\text{A20})$$

## 5. Reflection and Transmission Coefficients

The exact form of the reflection and transmission coefficients is found by imposing voltage and current continuity across the boundary between the RHTL and LHTL. This results in

$$\begin{aligned} T^\pm &= \Gamma^\pm + 1 \\ \frac{T^+}{Z_l} &= \frac{1}{Z_r^*} - \frac{\Gamma^+}{Z_r} \\ \frac{T^-}{Z_r} &= \frac{1}{Z_l^*} - \frac{\Gamma^-}{Z_l} \end{aligned}$$

Solving these two systems yields the reflection coefficients

$$\Gamma^+ = \frac{Z_l - Z_r^*}{Z_l + Z_r} \frac{Z_r}{Z_r^*}$$

$$\Gamma^- = \frac{Z_r - Z_l^*}{Z_l + Z_r} \frac{Z_l}{Z_l^*}.$$

They enjoy the property  $|\Gamma^+|^2 = |\Gamma^-|^2$ . Given that the CTL is a passive system energy flowing in the RHTL must be equal to energy flowing in the LHTL. Making sure of this serves as a consistency check. The time averaged power along a transmission line is  $P_{\text{av}}^\pm = \text{Re}[V_n^\pm I_n^{\pm*}]/2$ . It is important to note that we expect the outcome to be independent of the position along the CTL since there are no power sinks or sources. Thus, in terms of time averaged power, the following must hold

$$\frac{1}{2} \text{Re}[V_{n<0} I_{n<0}^*] = \frac{1}{2} \text{Re}[V_{n>0} I_{n>0}^*]$$

If the  $+$  solution is considered, i.e. voltage given by (A9) and current by (A19), then the energy flows in the RHTL and LHTL are respectively

$$P_{\text{av}}^{r+} = \frac{1}{2} \text{Re} \left[ \frac{1}{Z_r} - \frac{|\Gamma^+|^2}{Z_r} - \frac{\Gamma^{+*}}{Z_r^*} e^{ik_r n \Delta x} + \frac{\Gamma^+}{Z_r} e^{-ik_r n \Delta x} \right]$$

$$= \frac{2|Z_r|^2 \text{Re}(Z_l^*) + \text{Re}(Z_l(Z_r^{*2} + Z_r^2))}{2|Z_r|^2 |Z_r + Z_l|^2}$$

$$P_{\text{av}}^{l+} = \frac{1}{2} \text{Re} \left[ \frac{|T^+|^2}{Z_l^*} \right]$$

$$= \frac{2|Z_r|^2 \text{Re}(Z_l) + \text{Re}(Z_l(Z_r^{*2} + Z_r^2))}{2|Z_r|^2 |Z_r + Z_l|^2}.$$

The two terms with  $\Gamma^+ e^{-ik_r}/Z_r$  arise from the interference between incoming wave and reflected wave, however they do not contribute to the power since  $A - A^* \in i\mathbb{R}$ . By comparing these two equations it is immediately seen that  $P_{\text{av}}^{r+} = P_{\text{av}}^{l+}$ , i.e. energy is conserved. The same reasoning applies if the  $-$  solutions of voltage and current are considered.

#### a. Impedance Matching

For the two parts of the line to be able to communicate, the energy flow across the boundary must be greater than zero. Using  $Z_r^2 = L_r/C_r$  and writing  $P_{\text{av}}^+$  as function of frequency yields

$$P_{\text{av}}^+(\omega) = \frac{2\sqrt{\frac{L_l}{C_l} \left(1 - \frac{\omega_{\text{IR}}^2}{\omega^2}\right)}}{\frac{L_r}{C_r} + \frac{L_l}{C_l} + 2\sqrt{\frac{L_r L_l}{C_r C_l} \left(1 - \frac{\omega_{\text{IR}}^2}{\omega^2}\right)}}$$

The flow of power across the boundary is thus maximum when the impedances are matched in magnitude i.e.  $L_r/C_r = L_l/C_l$ . In this case and since the voltage is taken to have unit amplitude, the maximum power is  $(2Z_r)^{-1}$ . This corresponds to the limit  $\omega \rightarrow \infty$  for which the two lines are perfectly matched. Additionally and regardless of impedance, the following properties hold when going closer in frequency to the IR cutoff

$$\lim_{\omega \rightarrow \omega_{\text{IR}}} P_{\text{av}} = 0$$

$$\lim_{\omega \rightarrow \omega_{\text{IR}}} \frac{\partial P_{\text{av}}}{\partial \omega} = \infty$$

The power flow as function of frequency increases monotonically. The infinit slope of  $P_{\text{av}}(\omega)$  at  $\omega_{\text{IR}}$  guarantees that the first mode will have non-zero power flowing across the boundary. An example of the power across the boundary of a  $50 \Omega$  CTL is shown in Fig. 7. The black dots represent the normal modes, discussed in section A 7. The lowest mode obtained in a real geometry, with frequency  $\omega_1/2\pi = 4.149$  GHz has a power flow of 42.1% of the maximum despite the fact that  $P_{\text{av}}(\omega_{\text{IR}}) = 0$ .

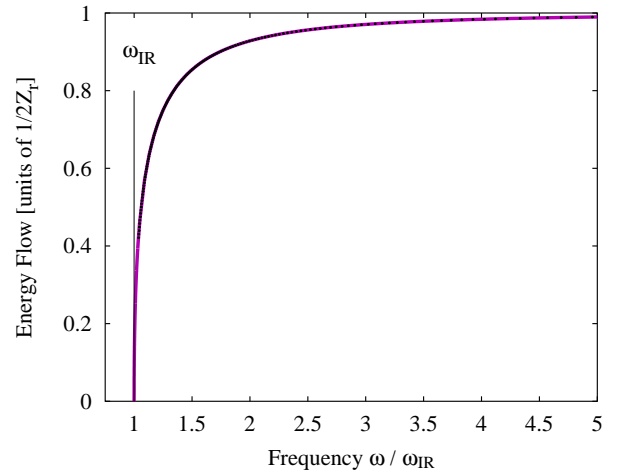


FIG. 7. Power flow across the boundary as function of mode frequency. The black dots are the numerically found modes discussed in Fig. 2(a) of the main text.

## 6. Voltage on the Composite Transmission Line

In the following boundary conditions are imposed on the voltage allowing a normal mode decomposition to be found. The normal mode indexed by  $k$  has a frequency  $\omega_k$  and wave vectors  $k_{r,l}(\omega_k)$ . The voltage along the line, for mode  $k$ , at node  $n$  and time  $t$  is given by a superposition of  $V^+$  and  $V^-$

$$V_{n,k}(t) = V_{n,k}^+(t) + V_0^{(k)} e^{i\Delta\phi_k} V_{n,k}^-(t).$$

The coefficients  $V_0^{(k)}$  and  $\Delta\phi_k$  are both real and are introduced so as to match the phases and amplitudes of

both waves. They depend on the frequency of the mode considered. The discussions in the last sections were restricted to the interface and in the lines but did not include effects linked to the finite length of the CTL. Here shall be considered the case where the line is asymmetric; the node in the right-handed line furthest away from the interface is labeled  $N_r$  and the one for the left-handed line

is  $-N_l \neq -N_r$ . The boundary conditions will determine what modes the CTL can support. Similarly to appendix A in [1], the boundary conditions for an open-circuited CTL are given by charge neutrality and impose that the spatial integral of the voltage vanish at the extremities of the TL. The integrals  $\theta^-$  and  $\theta^+$  are respectively found from equations (A9) and (A10)

$$\theta_n^+(t) = e^{-i\omega t} \begin{cases} -\frac{1}{ik_r \Delta x} e^{-ik_r n} + \frac{\Gamma^+}{ik_r} e^{ik_r n \Delta x} & n \leq 0 \\ \frac{T^+}{ik_l} e^{ik_l n \Delta x} & n \geq 0 \end{cases}$$

$$\theta_n^-(t) = e^{-i\omega t} \begin{cases} \frac{T^-}{ik_r} e^{ik_r n \Delta x} & n \leq 0 \\ -\frac{1}{ik_l} e^{-ik_l n \Delta x} + \frac{\Gamma^-}{ik_l} e^{ik_l n \Delta x} & n \geq 0 \end{cases}$$

At the extremity of the RHTL and LHTL  $\theta$  for mode  $k$  is respectively given by

$$\theta_{N_r, k} = \frac{1}{ik_r} \left[ -e^{-ik_r N_r \Delta x} + \Gamma^+ e^{ik_r N_r \Delta x} + V_0^{(k)} T^- e^{i(k_r N_r \Delta x + \Delta \phi_k)} \right] e^{-i\omega_k t}$$

$$\theta_{-N_l, k} = \frac{1}{ik_l} \left[ -V_0^{(k)} e^{i(k_l N_l \Delta x + \Delta \phi_k)} + V_0^{(k)} \Gamma^- e^{i(-k_l N_l \Delta x + \Delta \phi_k)} + T^+ e^{-ik_l N_l \Delta x} \right] e^{-i\omega_k t}.$$

The boundary conditions are  $|\theta_{-N_r, k}|^2 = |\theta_{N_l, k}|^2 = 0$ ; implying that a solution for the following non-linear system of equations has to be found

$$\begin{cases} 0 = \left| e^{-2ik_r N_r \Delta x} - \Gamma^+ - V_0^{(k)} T^- e^{i\Delta \phi_k} \right|^2 \\ 0 = \left| V_0^{(k)} e^{i2k_l N_l \Delta x} - V_0^{(k)} \Gamma^- - T^+ e^{-i\Delta \phi_k} \right|^2 \end{cases} \quad (\text{A21})$$

Where it is understood that  $\Gamma^\pm$ ,  $T^\pm$ ,  $k_r$ ,  $k_l$ ,  $V_0^{(k)}$  and  $\Delta \phi_k$  all depend on the mode frequency  $\omega_k$ . This system has four parameters  $C_{r,l}$  and  $L_{r,l}$  and three unknowns. It is solved numerically by selecting the desired line capacitance and inductance and searching for  $(V_0^{(k)}, \Delta \phi_k, \omega_k)$  triplets. As example, three modes supported by a 50  $\Omega$  CTL with  $c_r = 1667$  fF/ $\mu$ m,  $l_r = 4167$  pH/ $\mu$ m,  $C_l = 398$  fF and  $L_l = 995$  pH are shown in Fig. 2(b) of the main text. The LHTL is made of 200 cells and the RHTL is 3 cm long. The values for the capacitance and inductance were chosen so that the IR cutoff is at 4 GHz. The voltage at the unit cells is indicated by black dots and the coloured lines are to help visualizing the modes. As can be seen from the figure, the composite line can support modes with slightly different frequency. For these neighbouring modes the voltage maximums in the RHTL are only slightly shifted in position. This is the result of the wavelength in the LHTL being very small. A small change in frequency is thus enough to include and extra anti-node in the LHTL. For the high frequency modes this trend is reversed the wavelength is short in the RHTL and long in the LHTL. The similar voltage profile in the

RHTL at low frequencies is what will allow cQED to reach the multi-mode ultrastrong coupling regime.

## 7. Density of Modes

The density of modes (DOM) is found by numerically solving equations (A21). An example of DOM for a CTL is shown in Fig. 2(a) of the main text. A theoretical understanding can be gained by considering the right-handed and left-handed transmission lines separately. The DOM gives the number of modes within the interval  $k$  to  $k + dk$  by  $\mathcal{D}(k)dk = dn$ . The frequency dependent DOM is

$$\mathcal{D}(\omega) = \frac{dn}{d\omega} \frac{1}{dk}$$

In the case of a chain of  $N$  identical cells of length  $L = N\Delta x$  the mode numbered by  $n$  has a wave number  $k = n2\pi/L$  thus  $dn/dk = L/2\pi$ . Using the dispersion relations given by (A3) and (A4) the DOM for the individual systems are

$$\mathcal{D}_r(\omega) = \frac{\sqrt{C_r L_r}}{\pi}$$

$$\mathcal{D}_l(\omega) = \frac{4N_l \sqrt{C_l L_l}}{\pi} \tan \phi_l \sin \phi_l, \quad \phi_l = \frac{k_l(\omega) \Delta x}{2}$$

$N_l$  is the number of unit cells making up the LHTL. The DOM is then approximated by  $\mathcal{D}(\omega) = \mathcal{D}_r(\omega) + \mathcal{D}_l(\omega)$ . As shown by Fig. 2(a) of the main text there is a good



agreement between the approximate and the numerically exact DOM up to some small oscillations. Using the dispersion relations to make explicit the dependence on  $\omega$  yields

$$\mathcal{D}(\omega) = \frac{2N_l}{\pi} \frac{\omega_{\text{IR}}}{\omega \sqrt{\omega^2 - \omega_{\text{IR}}^2}} + \frac{d_r \sqrt{c_r l_r}}{\pi}$$

$d_r$  is the length of the RHTL taken to the continuum. In practice and close to the IR cutoff the last term, coming from the RHTL, can be neglected since the DoM is dominated by the LHTL. This is what we call a sub-sub-ohmic density of states. Indeed, expanding the functions in the DOM as a power law around  $\omega_{\text{IR}}$ . Expansion in  $\delta\omega = \omega - \omega_{\text{IR}}$  reveals that  $\mathcal{D}_l(\delta\omega) \propto (\delta\omega)^{-1/2}$ . Dropping second order terms in  $\delta\omega$  results in

$$\mathcal{D}(\omega_{\text{IR}} + \delta\omega) \approx \frac{N_l}{\sqrt{2}\pi} \frac{1}{\sqrt{\omega_{\text{IR}} \delta\omega}}$$

## Appendix B: Coupling a Qubit to the Composite Transmission Line

As shown in the section A 6, neighbouring modes with low frequencies have close spatial structures in the RHTL. Indeed in the LHTL a low frequency mode has a short wavelength; a small change in frequency is sufficient to produce an extra node. At high frequencies the situation is reversed; the wavelength is short in the RHTL and long in the LHTL. Given that for neighbouring low frequency modes in the RHTL, the anti-nodes are close together, it is interesting to put a qubit close to the CTL. Thus coupling it to many modes. The situation of multimode ultrastrong coupling arises when there are  $N > 1$  modes within an interval of  $2g_k$  and  $g_k/\omega_k > 0.1$ . This situation is realized with flux qubits [12]. The last condition corresponds to coupling strengths for which the rotating wave approximation starts breaking down. In some situations it can also be desired to have only strong coupling between the qubit in the CTL. In this case the qubit will still interact with many modes but the RWA will hold.

### 1. Qubit Size Factor

To couple a qubit to as many modes as possible, the anti-nodes must move as little as possible. This is achieved in the RHTL at low frequencies close to  $\omega_{\text{IR}}$ . Thus if  $\Delta_q$  is the qubit's transition frequency,  $\omega_{\text{IR}} \lesssim \Delta_0$  is desired. The effect of the qubit's size can be taken into

account through the finite size factor defined by

$$F_k = \frac{\int_Q dx V_k(x)}{\max_k \left\{ \int_Q dx V_k(x) \right\}}.$$

The integral of the voltage is taken over the qubit's spatial extent and normalized to the mode that couples strongest to the qubit. If the qubit should be placed at a current anti-node, such as flux qubits, then voltage is replaced by current in the definition of the size factor.  $F_k$  shows which modes couple the most to the qubit. Given the mode structure a qubit coupled in the RHTL (LHTL) will couple to many low (high) frequency modes. The case of a 0.5 mm long flux qubit coupled to the RHTL is plotted in Fig. 2(a) of the main text by the red line showing  $F_k \mathcal{D}(\omega_k)$ . Using this DOM and designing the flux qubit to couple to the 4.751 MHz mode with strength 200 MHz, results in ultrastrong coupling to 36 neighbouring modes within a  $\pm 200$  MHz range.

## 2. Multimode Ultrastrong Coupling and the Spin-Boson Model

It has been experimentally demonstrated that flux qubits can achieve the regime of ultrastrong coupling where the RWA breaks down [11, 12]. A theoretical survey of how this is achieved is described by Bourassa *et al.*[10]. The flux qubit has a double well potential; when coupled to a CTL the system is described by a Spin-Boson model

$$\hat{H}_{sb} = -\frac{1}{2}\Delta_0 \hat{\tau}_x + \frac{1}{2}\varepsilon \hat{\tau}_z + \sum_{k=1}^M \omega_k \hat{a}_k^\dagger \hat{a}_k + \hat{\tau}_z \sum_{k=1}^M g F_k \left( \hat{a}_k^\dagger + \hat{a}_k \right)$$

$\hat{a}_k$  and  $\hat{a}_k^\dagger$  are the creation and annihilation operators for mode  $k$  of the CTL. Here we have, compared to the convention in the rest of this work, rotated the basis for the two-state system according to  $\hat{\sigma}_{z/x} = \hat{\tau}_{x/z}$  in order to connect to the spin-boson literature. The coupling strength  $g_k$  is written as  $g F_k$  to be able to simultaneously tune the coupling to all the modes whilst keeping the effect of the qubit's size. In the following the symmetric case shall be assumed by setting the bias to zero, i.e  $\varepsilon = 0$ . When coupling the qubit to the modes close to the lower frequency limit  $\omega_{\text{IR}}$  the density of states is well below the sub-ohmic limit (in this case  $s = -1/2 < 0$ ). The coupling with the bosonic bath leads to a renormalization of the tunnelling element  $\Delta_0 \mapsto \Delta_{\text{eff}}$ .

### a. Approaching a quantum phase transition

Showing that the qubit-CTL system approaches a quantum phase transition up to finite size effects, is done by making use of the map between the Spin-Boson model and the 1D Ising chain [30]. A true quantum phase transition i.e.  $\Delta_{\text{eff}} = 0$  is reached only in an infinitely dense LHTL. Following the reasoning in Weiss 28 these partition functions are expressed as sums over the number  $m$

$$Z_{\text{sb}} = \sum_{\pm q_0} \sum_{m=0}^{\infty} \left(\frac{\Delta_0}{2}\right)^{2m} \int_0^{\infty} ds_{2m} \int_0^{s_{2m}} ds_{2m-1} \dots \int_0^{s_2} ds_1 \times \exp \left\{ \sum_{j=2}^{2m} \sum_{i=1}^{j-1} (-1)^{i+j} W(s_j - s_i) \right\}$$

$$Z_{\text{ising}} = \sum_{m=0}^{\infty} e^{-4m\beta U(0)} \int_0^{\hbar\beta - x_c} \frac{ds_{2m}}{x_c} \int_0^{s_{2m} - x_c} \frac{ds_{2m-1}}{x_c} \dots \int_0^{s_2 - x_c} \frac{ds_1}{x_c} \times \exp \left\{ 4 \sum_{j=2}^{2m} \sum_{i=1}^{j-1} (-1)^{j+i} U[(s_j - s_i)/x_c] \right\}$$

In the flux qubit-CTL system a kink corresponds to the system tunnelling from one side of the potential well to the other. In the 1D Ising chain a kink corresponds to a spin domain wall. By comparing the two partition functions, it is seen that the second integral of the Ising interaction potential  $U$  plays that same role as the kink-kink interaction of the Spin-Boson model  $W(r)$ . Therefore the quantity  $\partial_\tau^2 W(\tau)$  is equivalent to the interaction potential of the Ising model. Using the definition of  $W$  gives

$$\begin{aligned} V(\tau) &= \partial_\tau^2 W(\tau) = \alpha \partial_\tau^2 \int_0^\infty d\omega \frac{\mathcal{D}(\omega)}{\omega^2} (1 - e^{-\omega\tau}) \\ &= \frac{\alpha N_l}{\sqrt{2\omega_{\text{IR}}\pi}} \partial_\tau^2 \int_{\omega_{\text{IR}}}^\infty d\omega \frac{1 - e^{-\omega\tau}}{\omega^2 \sqrt{\omega - \omega_{\text{IR}}}} \\ &= -\frac{\alpha N_l}{\sqrt{2\omega_{\text{IR}}\pi}} \sqrt{\frac{\pi}{\tau}} e^{-\omega_{\text{IR}}\tau} \end{aligned}$$

The constant  $\alpha$  includes all the dependencies of qubit-CTL coupling. An example of which can be found in [10]. Here the density of states was approximated by the  $\omega^{-1/2}$  law valid close to the IR cutoff. The interaction potential has an exponential cutoff, and it is known that this model has no phase transition. In the Ising model this corresponds to forming domains of size  $1/\omega_{\text{IR}}$ . These domains then only possess nearest neighbour interaction, for which there is no phase transition. In the case of a flux qubit coupled to the CTL, the bare tunnelling element  $\Delta_0$  is renormalized until it is pushed below the IR cutoff. At this point, there are no more modes left to further reduce  $\Delta_{\text{eff}}$ .

#### b. Adiabatic Renormalization

It is expected that if the qubit is tuned so that  $\Delta_0 > \omega_{\text{IR}}$  the energy splitting will get renormalized to  $\Delta_{\text{eff}}$ . This process can be well approximated by adiabatic renormalization [6]. This process abruptly stops when  $\Delta_{\text{eff}} < \omega_{\text{IR}}$ . It is expected that the system is described by the state

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \bigotimes_k |\lambda_k\rangle \pm |1\rangle \bigotimes_k |-\lambda_k\rangle \right)$$

where the coherent states  $|\lambda_k\rangle$  and  $|-\lambda_k\rangle$  are created by the displacement operator  $\mathcal{D}(\lambda_k) = \exp\{\lambda_k \hat{a}_k^\dagger + \lambda_k^* \hat{a}_k\}$  with a sign depending on whether the qubit is in the  $|0\rangle$  or  $|1\rangle$  state. Adiabatic renormalization suggests that  $\lambda_k = g_k^2 \Theta(\omega - \Delta_{\text{eff}})/\omega_k^2$ . The set of modes with  $\omega_k > \Delta_0$  renormalize the tunnelling element

$$\Delta' = \Delta_0 \exp \left\{ -2 \sum_{\omega_k > \Delta_0} \frac{g_k^2}{\omega_k^2} \right\}.$$

This results in a new set of modes satisfying  $\Delta_0 > \omega_k > \Delta'$ . The latter formula can be iterated to find the fully renormalized tunnelling element  $\Delta_{\text{eff}}$ . Figure 4 shows the renormalized tunnelling element as function of the coupling strength. The finite size of the qubit was also taken into account according to section B1. As can be seen there is a sudden drop in  $\Delta_{\text{eff}}$  once some critical coupling strength  $g_{\text{crit}}$  is reached. The renormalized coupling element does not drop to zero but to some finite value as expected from Section B2a. The values of  $g_{\text{crit}}$  correspond to what is experimentally achievable [12] indeed, Niemczyk *et al.* have measured  $g/2\pi = 636$  MHz with  $\omega/2\pi = 5.357$  GHz. By identifying the regions where  $\Delta_{\text{eff}} < \omega_{\text{IR}}$  in the  $(\Delta_0, g)$  plane shows regions where this strong renormalization occurs. The border between the two regions is accompanied by a discrete jump in  $\Delta_{\text{eff}}$  as shown by Fig. 4 of the main text.

### Appendix C: Physical Realization

For the CTL to reach the quantum regime it should be superconducting [5]. Here we propose to fabricate the coupled transmission lines by standard fabrication of the RHTL and using Josephson Junction Arrays (JJA) to create the LHTL as Josephson junctions provide a compact way to engineer sizeable inductances [32–34]. These Josephson junctions are large enough to be used in the linear regime only. The values of the critical current and capacitance are determined by impedance matching between the two parts of the line to ensure maximum transmission, and the desired IR cutoff. The impedance of the RHTL is  $Z_r = \sqrt{l_r/c_r}$  and should be chosen to be as close as possible to  $Z_0 = 50 \Omega$  to match the impedance of the electronics. For generating multi-mode entangled states and studying the Spin-Boson model, the IR cutoff frequency should be close to the qubit transition fre-

quency  $\Delta_0$  and the impedance  $|Z_l|$  close to  $Z_0$ . Thus using impedance matching and the definition of  $\omega_{\text{IR}}$ , i.e. equation (A5), the parameters of the LHTL unit cell are

$$C_l = \frac{1}{2\omega_{\text{IR}}Z_0} \quad \text{and} \quad L_l = \frac{Z_0}{2\omega_{\text{IR}}}$$

Most superconducting qubit transition frequencies are in the range 4 to 12 GHz placing the capacitance in the range  $C_l \in 398$  fF to 133 fF. The inductance should be from 995 pH to 332 pH respectively. To first order the inductance of a JJ is

$$L_l = \frac{\hbar}{2eI_0}.$$

A desired inductance in the range 995 pH to 332 pH corresponds roughly to a critical current in the range  $I_c \in 330$  to 990 nA. For a fixed cross section and material, the length of the RHTL has the effect of changing the fundamental frequency. For the continuous RHTL alone the frequencies of the normal modes are  $\omega_k = k\pi/\sqrt{c_r l_r} d_r$  where  $c_r$  and  $l_r$  are the capacitance and inductance per unit length respectively. As an example it can be desired to have a voltage anti-node in the middle of the RHTL. This sets the capacitance and inductance per unit length to

$$c_r = \frac{2\pi}{\omega_{\text{IR}}Z_0 d_r} \quad \text{and} \quad l_r = \frac{2\pi Z_0}{\omega_{\text{IR}} d_r}$$

An IR cutoff at 4 GHz and a 3 cm long RHTL sets  $c_r = 1667$  fF/ $\mu\text{m}$  and  $l_r = 4167$  pH/ $\mu\text{m}$ .

### 1. Parasitic Capacitances due to the JJ's

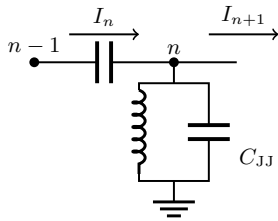


FIG. 8. Modified left-handed unit cell taking into account the capacitance of the JJ.

The Josephson Junctions making up the inductors of the LHTL additionally have a capacitance in parallel to the inductor. The unit cell is thus more properly modelled by the one shown in Fig. 8. Taking this capacitor into account, the Lagrangian of the LHTL now reads

$$\mathcal{L}_l = \sum_{n < 0} \frac{1}{2L_l} \Phi_n^2 - \frac{1}{2} C_{\text{JJ}} \dot{\Phi}_n^2 - \frac{1}{2} C_l (\dot{\Phi}_n - \dot{\Phi}_{n+1})^2$$

The equation of motion for which is

$$C_l (\ddot{\Phi}_{n-1} - 2\ddot{\Phi}_n + \ddot{\Phi}_{n+1}) - C_{\text{JJ}} \ddot{\Phi}_n = \frac{1}{L_l} \Phi_n$$

The ansatz  $\Phi_n(t) = \Phi_0 \exp\{-i(k_l n \Delta x - \omega t)\}$  produces the dispersion relation

$$\omega_l(k_l) = \frac{1}{2\sqrt{C_l L_l}} \left[ \sin^2 \left( \frac{k_l \Delta x}{2} \right) - \frac{C_{\text{JJ}}}{4C_l} \right]^{-1/2}$$

At the infra-red cutoff  $\sin(k\Delta/2) = 1$  thus as long as  $C_{\text{JJ}} \ll 4C_l$  the influence of the JJ's capacitor will be negligible.

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